# A discussion on a possibility to interpret quantum mechanics in terms of general relativity

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**Abstract**

It is shown that, with some reasonable assumptions, the theory of general relativity can be made compatible with quantum mechanics by using the field equations of general relativity to construct a Robertson-Walker metric for a quantum particle so that the line element of the particle can be transformed entirely to that of the Minkowski spacetime, which is assumed by a quantum observer, and the spacetime dynamics of the particle described by a Minkowski observer takes the form of quantum mechanics. Spacetime structure of a quantum particle may have either positive or negative curvature. However, in order to be describable using the familiar framework of quantum mechanics, the spacetime structure of a quantum particle must be "quantised" by an introduction of the imaginary number $i$. If a particle has a positive curvature then the quantisation is equivalent to turning the pseudo-Riemannian spacetime of the particle into a Riemannian spacetime. This means that it is assumed the particle is capable of measuring its temporal distance like its spatial distances. On the other hand, when a particle has a negative curvature and a negative energy density then quantising the spacetime structure of the particle is equivalent to viewing the particle as if it had a positive curvature and a positive energy density.

It has been considered that general relativity may not be compatible with the quantum theory because the former is formulated in terms of curved spacetimes while the later is based on the view of an observer who uses the Minkowski spacetime and describes the quantum dynamics of a particle in terms of a Hilbert space of physical states. However, it can be seen that the two descriptions may reconcile if a curved spacetime of a quantum particle can be transformed to the Minkowski spacetime so that a Minkowski formulation of the spacetime dynamics of the particle is that of the quantum theory. The following is a discussion of such a possibility.

Consider first the following situation [1]. If the dynamics of a particle is assumed to be described by the field equations of general relativity

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu} \tag{1}$$

then with $\Lambda = 0$ and a centrally symmetric spacetime metric [2]

$$ds^2 = e^{\mu}dt^2 - e^{\nu}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

and an energy-momentum tensor of the form

$$T_\mu^\nu = \begin{pmatrix} -\frac{\alpha\beta}{\kappa}\frac{e^{-\beta r}}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{\alpha\beta}{\kappa}\frac{e^{-\beta r}}{r^2} & 0 & 0 \\ 0 & 0 & \frac{\alpha\beta^2}{2\kappa}\frac{e^{-\beta r}}{r} & 0 \\ 0 & 0 & 0 & \frac{\alpha\beta^2}{2\kappa}\frac{e^{-\beta r}}{r} \end{pmatrix}, \tag{3}$$

the field equations of general relativity will admit as an exact solution the following line element of Yukawa potential form

$$e^{-\nu} = 1 - \alpha\frac{e^{-\beta r}}{r} + \frac{Q}{r}, \tag{4}$$

where the term $Q/r$ could be interpreted as the Coulomb repulsion force of a proton. Because the physical system is expressed in terms of curved spacetime of the particle, whether this solution is physically acceptable, although it is mathematically acceptable, could not be justified under the view of an observer who uses the Minkowski spacetime, unless the above curved spacetime of the particle can be transformed entirely to a manifestly Minkowski metric. However, the above form of the energy-momentum tensor seems to reveal a possibility that at the quantum level the energy density may vary only as an inverse square of distance and the pressure may be ignored compared to the energy density. With these observations, let us now consider a general relativistic spacetime model for a quantum particle using the Robertson-Walker metric [3, 4]

$$ds^2 = c^2dt^2 - S^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{5}$$

with the energy-momentum tensor $T_{\mu\nu}$ of the form

$$T_\mu^\nu = \begin{pmatrix} \frac{A}{S^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

The field equations of general relativity then reduce to the system

$$\frac{\dot{S}^2}{S^2} + \frac{kc^2}{S^2} - \frac{\Lambda c^2}{3} = \frac{\kappa c^2}{3}\frac{A}{S^2} \tag{7}$$

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2} + \frac{kc^2}{S^2} - \Lambda c^2 = 0. \tag{8}$$

This system of equations has a static solution

$$S_0^2 = \frac{kc^4}{4\pi G\epsilon} \tag{9}$$

which is similar to the Einstein static model with an energy density $\epsilon$ which may be very large [5]. However, in this case, the possibility of negative energy density should not be ruled out because at the quantum level a particle may have a curved spacetime with negative curvature as will be discussed in the following. Since we are discussing curved spacetimes at the quantum level, the quantity $\Lambda = k/S_0^2$ will change drastically for a small fraction of variation of $S_0$. Hence, the quantity $\Lambda$ should also be considered as an inverse square function of $S$, i.e. $\Lambda = B/S^2$, where $B$ is constant (actually the constant $B$ could be set to zero for almost all arguments that follow from here). The above system of field equations is then modified to the system of equations

$$\frac{\dot{S}^2}{S^2} + \frac{kc^2}{S^2} - \frac{c^2}{3}\frac{B}{S^2} = \frac{\kappa c^2}{3}\frac{A}{S^2} \tag{10}$$

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2} + \frac{kc^2}{S^2} - c^2\frac{B}{S^2} = 0. \tag{11}$$

This system of equations has a solution of the form

$$S = act, \qquad \text{where} \qquad a = \sqrt{\frac{\kappa A}{2} - k} \tag{12}$$

Let us first consider the case $k = 1$. It is seen that in this case a real solution requires spacetime of a quantum particle to have a very large positive energy density in its own reference frame. However, a particle with this large positive energy density can not be viewed by an observer who uses the Minkowski spacetime because the curved spacetime of the particle in this case can not be transformed to the Minkowski spacetime of a quantum observer. Furthermore, such a large energy density is not appropriate for the surrounding spacetime of quantum particles like protons and neutrons. Therefore, if we assume a

reasonable value for the energy density so that $\kappa A/2 \ll 1$, then we are forced to "quantize" the spacetime structure of the particle by introducing the imaginary number $i$ and let $a \approx i$ or $S \approx ict$. Actually, this kind of quantisation turns the pseudo-Riemannian curved spacetime of the particle into a Riemannian spacetime. This means that we assume the particle to be able to measure its temporal distance in exactly the same way as its spatial distances. The quantisation is realisable only when the curved spacetime of the particle can be viewed in the Minkowski spacetime. This is in fact the case for if we apply the coordinate transformations [3]

$$iR = ctr, \qquad cT = ct\sqrt{1-r^2} \tag{13}$$

then, as can be verified, these coordinate transformations reduce the Robertson-Walker metric of the quantum particle to a manifestly Minkowski metric of the form

$$ds^2 = c^2dT^2 - dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{14}$$

It is seen that the dynamics of spacetime structure of the quantum particle can now be investigated by an observer whose uses a Minkowski metric. The investigation can be carried out by writing the quantity $S$ in terms of the coordinates $(R, cT)$ in the form of an action integral

$$S = -i\sqrt{c^2T^2 - R^2} = -i\int ds = -ic\int\sqrt{1-\frac{v^2}{c^2}}dT \tag{15}$$

where $ds$ is the usual Minkowski spacetime interval and $v = R/T$. In addition, if we define a new quantity $\Psi$ by the relation $S = K\ln\Psi$, then we have

$$\Psi = e^{\frac{i}{K}\int ds}. \tag{16}$$

With this form, the familiar quantum mechanics can be recovered by applying the Feymann path integral method [6]. However, since in Minkowski spacetime the quantity $S$ has an action integral form, we can construct a quantum mechanics by following Schrödinger's method [7] as in his original derivation of the wave equation of quantum mechanics by observing that the quantity $S$ satisfies the relation

$$-\frac{1}{c^2}\left(\frac{\partial S}{\partial T}\right)^2 + \left(\frac{\partial S}{\partial R})\right)^2 - 1 = 0. \tag{17}$$

The quantity $\Psi$ then satisfies the relation

$$\frac{1}{c^2}\left(\frac{\partial \Psi}{\partial T}\right)^2 - \left(\frac{\partial \Psi}{\partial R}\right)^2 - \frac{1}{K^2}\Psi^2 = 0. \tag{18}$$

Since $\partial\Psi/\partial R = \nabla\Psi.\partial\mathbf{R}/\partial R = |\nabla\Psi|\cos\alpha$, using the variational principle, after averaging the above equation with $< \cos^2\alpha >= 1/2$, we obtain a Klein-Gordon-like wave equation [8]

$$-\frac{1}{c_a^2}\frac{\partial^2\Psi}{\partial T^2} + \nabla^2\Psi - \frac{1}{K_a^2}\Psi = 0, \tag{19}$$

where $c_a = c/\sqrt{2}$ and $K_a = K/\sqrt{2}$. If we compare this equation with the Klein-Gordon equation in quantum mechanics, then we see that this equation describes a quantum dynamics of a particle with an average velocity $c_a$, rather than that of light $c$. This may be a reason why the force carriers in strong and weak interactions have mass. The comparison also gives $K_a = \hbar/mc_a$. Here, perhaps, the most important point that should be emphasised is that the Minkowski coordinates in this case depend entirely on the metric structure of a quantum particle. So observers who use the Minkowski spacetime can not perform measurements of physical observables of the particle by their own choices of gauges of space and time. This may be the reason for unpredictable behaviours of quantum particles in the Minkowski quantum mechanics.

Now let us consider the case $k = -1$. Similar to the previous discussions, however, in this case we assume $a \approx 1$ or $S \approx ct$. The coordinate transformations of the form [3]

$$R = ctr, \qquad cT = ct\sqrt{1 + r^2} \tag{20}$$

also reduce the Robertson-Walker metric of a quantum particle to that of the Minkowski spacetime. The quantity $S$ written in terms of the coordinates $(R, cT)$ takes the form

$$S = \sqrt{c^2T^2 - R^2} = \int ds = c\int \sqrt{1 - \frac{v^2}{c^2}}dT \tag{21}$$

and satisfies the relation

$$-\frac{1}{c^2}\left(\frac{\partial S}{\partial T}\right)^2 + \left(\frac{\partial S}{\partial R}\right)^2 + 1 = 0. \tag{22}$$

The quantity $\Psi$ now becomes

$$\Psi = e^{\frac{1}{K}\int ds} \tag{23}$$

and satisfies the equation

$$-\frac{1}{c_a^2}\frac{\partial^2 \Psi}{\partial T^2} + \nabla^2\Psi + \frac{1}{K_a^2}\Psi = 0, \tag{24}$$

This equation differs from the Klein-Gordon equation by the plus sign before the last term. This results from the fact that the quantum particle in this case has negative curvature. This kind of structure of a particle is not assumed in quantum mechanics. However, it is seen that if the energy density in this case is negative then every thing up to this stage is real and compatible with the usual relativistic description with timelike intervals. If we want to "quantise" spacetime structure by turning the above equation into the Klein-Gordon equation so that we can describe the quantum dynamics of the particle using the familiar quantum mechanics then we can let $K \to iK$. This process of quantisation is equivalent to turning spacetime structure of a particle with negative curvature into that with positive curvature and specifying a positive energy density for the particle in the Minkowski spacetime.

Finally consider the case $k = 0$. In this case a real solution is obtained for any positive energy density, $S = act$. If we apply the coordinate transformations

$$R = actr, \qquad cT = ct \tag{25}$$

then $dR = actdr + acrdT$. It is seen that when the term $acrdT \ll 1$, the spacetime structure of a particle can be reduced to that of the Minkowski spacetime. A particle with large energy density, i.e. $a$ large, can only appear to a Minkowski observer for a short time $dT$. On the other hand, a particle with low energy density can exist with respect to a Minkowski observer for a long period of time $dT$.

As a conclusion we make some remarks about the energy-momentum conservation laws in general relativity. For simple models that have been discussed the only solution $S = S_0$ satisfies strictly the conservation laws required by the general theory. However, when the theory is applied into quantum physics the conservation laws should not be expected to satisfy at the quantum level, and the uncertainty principle would allow any such violation. This can also be seen by spacetime coordinate relationships discussed above. Furthermore, it seems that at the quantum level concepts like positiveness of energy density also become relative and coordinate-dependent, and this may affect the foundations of physics of a Minkowski observer. These sophisticated problems require further investigations.

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